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ARISTOTLE'S PHILOSOPHY OF MATHEMATICS

Jonathan Lear

The fundamental problem in the philosophy of mathematics, which has persisted from Plato's day until ours, is to provide an account of mathematical truth that is harmonious with our understanding of how we come to know mathematical truths.¹ In *Physics* B2 and *Metaphysics* M3 Aristotle provided the seeds of a unified philosophy of mathematics. This has not been generally appreciated for two reasons. First, it is commonly assumed that Aristotle thought that the objects in the natural world do not perfectly instantiate mathematical properties: a physical sphere is not truly spherical; a straight edge is not truly straight.² In consequence, though commentators see Aristotle as railing against a Platonic ontology of geometrical and arithmetical objects, they see him as unable to offer a genuinely alternative epistemology. "Mathematicians," according to one influential interpretation of Aristotle, "treat objects which are different from all sensible things, perfectly fulfill given conditions and are apprehensible by pure thought."³ This interpretation must view Aristotle as caught in the middle of a conjuring trick: trying to offer an apparently Platonic account of mathematical knowledge while refusing to allow the objects that the knowledge is knowledge of. Second, Aristotle's philosophy of mathematics is often labeled

¹ *Meno* 73e-87c, *Republic* 507e-527d. cf. Paul Benacerraf, "Mathematical Truth," *Journal of Philosophy*, 70 (1973); Kurt Gödel, "What is Cantor's Continuum Problem?" in *Philosophy of Mathematics: Selected Readings*, ed. Paul Benacerraf and Hilary Putnam (Englewood Cliffs: Prentice Hall, 1964); Charles Parsons, "Mathematical Intuition," *Proceedings of the Aristotelian Society*, 80 (1979-80); Penelope Maddy, "Perception and Mathematical Intuition," *Philosophical Review*, 89 (1980).

² Ian Mueller, "Aristotle on Geometrical Objects," *Archiv für Geschichte der Philosophie*, 52 (1970); Julia Annas, *Aristotle's Metaphysics M and N* (Oxford: Clarendon Press, 1976), p. 29. The literature in which this assumption is explicit or implicit is enormous. I mention two excellent works which, I think, provide the best defense of an interpretation with which I wish to disagree.

³ Mueller, op. cit., p. 157.

“abstractionist”—mathematical objects are formed by abstracting from the sensible properties of objects—and it is thought that he falls victim to Frege’s attacks on “abstractionist” philosophies of mathematics.⁴

However, Aristotle’s abstractionism is of enduring philosophical interest. For not only does it differ fundamentally from the psychologistic theories that Frege scorned; it represents a serious attempt to explain both how mathematics can be true and how one can have knowledge of mathematical truths.

I

In *Physics* B2 Aristotle begins to define mathematical activity by contrasting it with the study of nature:

The next point to consider is how the mathematician differs from the physicist. Obviously physical bodies contain surfaces, volumes, lines, and points, and these are the subject matter of mathematics Now the mathematician, though he too treats of these things (viz., surfaces, volumes, lengths, and points), does not treat them *as* the limits of a physical body; nor does he consider the attributes indicated as the attributes *of* such bodies. That is why he separates them, for in thought they are separable from motion (*kinēsis*), and it makes no difference nor does any falsity result if they are separated. Those who believe the theory of the forms do the same, though they are not aware of it; for they separate the objects of physics, which are less separable than those of mathematics. This is evident if one tries to state in each of the two cases the definitions of the things and of their attributes. *Odd, even, straight, curved*, and likewise *number, line, and figure* do not involve change; not so *flesh and bone and man*—these are defined like *snub nose*, not like *curved*. Similar evidence is supplied by the more physical branches of mathematics, such as optics, harmonics, astronomy. These are in a way the converse of geometry. While geometry investigates physical lengths, but not *as* physical, optics investigates mathematical lengths, but *as* physical, not *as* mathematical. [*Physics* B2, 193b23–194a12; my emphasis]

⁴Annas, *op. cit.*, pp. 31–33.

Several fundamental features of Aristotle's philosophy of mathematics emerge quite clearly from this passage.

- (1) Physical bodies do actually contain the surfaces, lengths, and points that are the subject matter of mathematics (193b23–25).
- (2) The mathematician does study the surfaces, volumes, lengths, and points of physical bodies, but he does not consider them *as* the surfaces, etc., *of* physical bodies (192b31–33). Geometry does investigate physical lengths, but not *as* physical (194a9–11).
- (3) The mathematician is able to study surfaces, volumes, lengths, and points in isolation from their physical instantiations because (in some way which needs to be explained) he is able to separate the two in thought (193b33 ff.).
- (4) Having been separated by thought, mathematical objects are free from the changes which physical objects undergo (193b34).
- (5) (For some reason which needs to be explained) no falsity results from this separation (193b34–35).

Indeed, it seems that it is only those things which the mathematician separates which can be legitimately separated. Platonists, in their so-called discovery of the Forms, make at least two mistakes. First, they do not realize that, at base, they are doing no more than engaging in this process of separation in thought (193b35). Second, they choose the wrong things to separate. The reason—and this must provide a significant clue to how Aristotle thought legitimate acts of separation could occur—seems to be that Platonists tried to separate from matter things that could not be conceived of except as enmattered. His paradigm contrast is that of *snub* with *curved*: an account of snubness cannot merely specify a shape; *snub* must be a shape embodied in a nose.⁵ Because it is necessarily enmattered, a snub thing cannot be conceived of as independent of physical change as, for example, a

⁵Cf., e.g., *Metaphysics* 1025b31; 1030b29 ff.; 1035a26; 1064a23; 1030b17; 1035a5; 1064a25.

curve can be. It is thus clear that Aristotle allows that there is *some* legitimate type of separation, differing from the Platonist separation of the Forms, such that if we understand how this separation occurs and why it is legitimate, we will understand how mathematics is possible.

The heart of Aristotle's philosophy of mathematics is presented in *Metaphysics* M3:

Just as universal propositions in mathematics are not about separate objects over and above magnitudes and numbers, but are about these, only not *as* having magnitude or being divisible, clearly it is also possible for there to be statements and proofs about (*peri*) perceptible magnitudes, but not *as* perceptible but *as* being of a certain kind. For just as there are many statements about things merely *as* moving apart from the nature of each such thing and its incidental properties (and this does not mean that there has to be either some moving object separate from the perceptible objects, or some such entity marked off in them), so in the case of moving things there will be statements and branches of knowledge about them, not *as* moving but merely *as* bodies, and again merely *as* planes and merely *as* lengths, *as* divisible and *as* indivisible but with position and merely *as* indivisible. So since it is true to say without qualification not only that separable things exist but also that nonseparable things exist (e.g., that moving things exist), it is also true to say without qualification that mathematical objects exist and are as they are said to be. It is true to say of other branches of knowledge, without qualification, that they are of this or that—not what is incidental (e.g., not the white, even if the branch of knowledge deals with the healthy and the healthy is white) but what each branch of knowledge is of, the healthy ⟨if it studies its subject⟩ *as* healthy, man if ⟨it studies it⟩ *as* man. And likewise with geometry: the mathematical branches of knowledge will not be about perceptible objects just because their objects happen to be perceptible, though not ⟨studied⟩ *as* perceptible; but neither will they be about other separate objects over and above these. Many properties hold true of things in their own right as being each of them of a certain type—for instance, there are attributes peculiar to animals as being male or as being female (yet there is no female or male separate from animals). So there are properties holding true of things merely *as* lengths or *as* planes.

The more what is known is prior in definition, and the simpler, the greater the accuracy (i.e., the simplicity) obtained. So there is

more accuracy where there is no magnitude than where there is, and most of all where there is no movement; though if there is movement accuracy is greatest if it is primary movement, this being the simplest, and uniform movement the simplest form of that.

The same account applies to harmonics and optics; neither studies its objects *as* seeing or *as* utterance, but *as* lines and numbers (these being proper attributes of the former); and mechanics likewise.

So if one posits objects separated from what is incidental to them and studies them *as* such, one will not because of this speak falsely any more than if one draws a foot on the ground and calls it a foot long when it is not a foot long; the falsehood is not part of the premises.

The best way of studying each thing would be this: to separate and posit what is not separate, as the arithmetician does and the geometer. A man is one and indivisible *as* a man, and the arithmetician posits him *as* one indivisible, then studies what is incidental to the man *as* indivisible; the geometer, on the other hand, studies him neither *as* a man nor *as* indivisible, but *as* a solid object. For clearly properties he would have had even if he had not been indivisible can belong to him irrespective of his being indivisible or a man [*aneu toutōn*]. That is why the geometers speak correctly: they talk about existing things and they really do exist—for what exists does so in one of two senses, in actuality or materially. [*Met.* M3, 1077b18–1078a31; my emphasis]

The point of the argument is to show that, *pace* Plato, one can allow that the mathematical sciences are true without having to admit the existence of ideal objects. This chapter follows a sustained critique of mathematical Platonism, and it should be read as possessing its own dialectical strategy, directed if not against Plato, then against Academic Platonists.⁶ Thus Aristotle begins with cases which he thinks either are or should be most embarrassing to the Platonist. By “the universal propositions in mathematics” (*ta katholou en tois mathēmasin*), he is probably referring to the general propositions expressed by Eudoxus’ theory of proportion.⁷ Aristotle reports that the fact that proportions alternate

⁶Cf. *Metaphysics* M2.

⁷Cf. *Metaphysics* 1077b17; Euclid, *Elementa* (Leipzig: B.G. Teubner, 1969–73), II, 280; and W. D. Ross, *Aristotle’s Metaphysics* (Oxford: Clarendon Press, 1975), II, 415.

—that if $a:b::c:d$, then $a:c::b:d$ —used to be proved separately for numbers, lines, solids, and times (*An. Pst.* A5, 74a19–25). Now, he says, it is proved universally (74a24). I suspect that the proof with which Aristotle was familiar differed slightly from Euclid V–16. For Euclid V presents a generalized theory of proportional *magnitudes*, but in *Metaphysics* M2, Aristotle explains his objection to Platonism on the basis of the universal propositions of mathematics as follows:

There are some universal propositions proved by mathematicians whose application extends beyond these objects [sc., Platonic numbers and geometrical objects]. So there will be another type of object here, between and separate from both Forms and intermediate objects, neither number nor point nor *magnitude* nor time. If this is impossible, clearly it is also impossible for the former [sc., numbers and geometrical objects] to exist in separation from perceptible objects. [*Met.* M2, 1077a9–14; my emphasis]

Thus the proof with which Aristotle was familiar probably had a slightly more algebraic character than Euclid V–16. The example of universal propositions in mathematics is supposed to cast doubt on the Platonists' move from the fact that there are sciences of arithmetic and geometry which issue arithmetical and geometrical truths, to the existence of Platonic numbers, planes, and solids.⁸ Aristotle's objection is *ad hominem*. If one believes that the mathematical sciences guarantee the existence of Platonic numbers and geometrical objects, then one should also believe in the existence of ideal objects which the generalized science of proportion is about. But neither are there any obvious

⁸At *Metaphysics* 1079a8–11 Aristotle says “according to the argument from the sciences there will be forms of everything of which there is a science.” The exact structure of the argument from the sciences is a complex and difficult problem, which lies beyond the scope of this paper. For those who wish to reconstruct it, cf. *Metaphysics* 990b8–17, 987a32–b18, *Peri Ideōn*, fragments 3 and 4 in W.D. Ross, *Aristotelis Fragmenta Selecta* (Oxford: Clarendon Press, 1974). See also M. Hayduck's text of Alexander's commentary on the noted passage in *Metaphysics* A9, *In Aristotelis Metaphysica Commentaria*, (Berlin, 1891). And cf. *Republic* 479a–480a, *Timaeus* 51d–52a. For recent commentary see, e.g., W.D. Ross, op. cit., I, pp. 191–93; H. Cherniss, *Aristotle's Criticism of Plato and the Academy* (New York: Russell and Russell, 1964), pp. 226–60, 272–318, 488–512. An even more difficult question, which also lies beyond the scope of this paper, is: to what extent, if any, did Aristotle himself accept a, perhaps restricted, version of the argument from the sciences? See note 27 below.

candidates, as squares, circles, and numbers are obvious candidates for geometry and arithmetic, nor would the Platonist be happy to “discover” a new ideal object. For the theory of proportion is applicable to numbers, lines, solids, and times, though it is not about any particular objects over and above these.

Immediately before discussing the various distinct proofs that proportions alternate, Aristotle says that occasionally we cannot grasp the full universality of a truth because there is no name that encompasses all the different sorts of objects to which the truth applies (*An. Pst.* A5, 74a8). In these passages, Aristotle uses the word “*megethos*” to refer strictly to spatial magnitudes; whereas Euclid’s use of “*megethos*” in *Elements* Book V is best understood as being the general term Aristotle sought after, applicable to spatial magnitudes, numbers, and times. (For this broad Euclidean sense of “*megethos*” I shall use the term “magnitude”; and I shall reserve “spatial magnitude” for the Aristotelian “*megethos*”.) So Aristotle’s point at the beginning of *Metaphysics* M3 can now be summed up as follows: the generalized theory of proportion need not commit us to the existence of any special objects—magnitudes—over and above numbers and spatial magnitudes. The theory is about spatial magnitudes and numbers, only not *as* spatial magnitude or number, but rather *as* magnitude: that is, they exhibit a common property, and they are being considered solely in respect of this.

The second difficulty for the Academic Platonist to which Aristotle alludes is presented by astronomy.⁹ Aristotle clearly thought that the Platonist should feel embarrassed by having to postulate a heaven separate from the perceptible heaven and also by having to admit that ideal objects move.¹⁰ The actual planets themselves were, Aristotle thought, eternal and unchanging (in the relevant respects) and so could function as the objects that the unchanging truths of astronomy were about. It seems that Aristotle thought he could get his Platonist opponent—perhaps a philosopher who is already retreating from Platonism,

⁹Cf. *Metaphysics* 1077b23 ff.; 1077a1–5; 997a84–99a19. See also the references to astronomy in *Physics* 193b24–194a12, quoted in part above.

¹⁰However, it is far from clear that Plato was embarrassed by this. Cf. *Republic* 528a–b, 529c–d. Perhaps the movement of ideal objects became an embarrassment to the Academic Platonists.

but not yet sure how far to retreat—to admit that astronomy is about the heavenly planets, but considered solely in terms of their properties as moving bodies.

But having come this far, the retreating Platonist cannot stop here. For if one admits that there are physical objects that can be treated solely *as* moving bodies, in isolation from all their other particular properties, then one can treat these objects solely *as* bodies, solely *as* planes, lengths, and so on (1077b23–30). Reality, Aristotle seems to be saying, can be considered under various aspects. Given the paradigm Aristotelian substances—individual men, horses, tables, planets—we are able to consider certain features of these substances in isolation.

Generalizing, one might say that Aristotle is introducing an *as*-operator, or *qua*-operator, which works as follows. Let *b* be an Aristotelian substance and let “*b qua F*” signify that *b* is being considered *as an F*. Then a property is said to be true of *b qua F* if and only if *b* is an *F* and its having that property follows of necessity from its being an *F*.¹¹

$$G(b \text{ qua } F) \leftrightarrow F(b) \ \& \ (F(x) \vdash G(x)).$$

Thus to use the *qua*-operator is to place ourselves behind a veil of ignorance: we allow ourselves to know only that *b* is *F* and then determine on the basis of that knowledge alone what other properties must hold of it. If, for example, *b* is a bronze isosceles triangle—

$$Br(b) \ \& \ Is(b) \ \& \ Tr(b) \text{—}$$

then to consider *b* as a triangle—*b qua Tr*—is to apply a predicate filter: it filters out the predicates like *Br* and *Is* that happen to be true of *b*, but are irrelevant to our current concern.

The filter enables Aristotle to make a different use of the distinction between incidental (*kata sumbebēkos*) and essential (*kath'*

¹¹I use the turnstile here to signify the relation “follows of necessity,” which I argue elsewhere Aristotle took as primitive. “ $X \vdash P$ ” should be read “*P* follows of necessity from *X*.” Cf. *Prior Analytics* 25b28–31 and my *Aristotle and Logical Theory* (Cambridge and New York: Cambridge University Press, 1980), Chapter 1.

hautō) predication from that which is often attributed to him.¹² On the standard interpretation, one is given an Aristotelian substance—for example, an individual man—and the essential predicates are those that must hold of the object if that object is to be the substance that it is. They are what it is for the substance to be the substance that it is. An incidental predicate is one that happens to hold of the substance, but is such that if it failed to do so the substance would continue to exist. For Aristotle, “rational” would be an essential predicate of Socrates; “snub-nosed” and “pale” would be incidental.

In *Metaphysics* M3, however, the distinction between essential and incidental predicates can be made only for an object *under a certain description*. If we are considering *b* as being an *F*, then every predicate that is not essential to its being *F* is considered incidental, even though it may be essential to *b*'s being the substance that it is.¹³ That is why in the definition of “*G(b qua F)*” one should have “(*F(x) ⊢ G(x)*)” on the right hand side of the equivalence rather than “*F(b) ⊢ G(b)*.” For one might have *F(b) ⊢ G(b)* in virtue of what *b* is instead of in virtue of what *F* and *G* are.

For example, Aristotle thought that the heavenly bodies must be composed of a special stuff different from and more divine than earth, air, fire, or water; he also thought that the heavenly bodies must be indestructible (*De Caelo* A2, 10). However, if one takes an arbitrary star and applies the predicate filter so that one considers it solely *as a sphere*, all the properties that do not follow from its being a sphere (its being composed of special stuff, its indestructibility, etc.) are *from this perspective* incidental. By applying a predicate filter to an object instantiating the relevant geometrical property, we will filter out all predicates which concern the material composition of the object.¹⁴ Thus the geometer is able to study perceptible material objects—this is indeed all that he studies—but he does not study them *as* perceptible or *as* material.

¹² Cf. *Metaphysics* 1077b34–1078a9; 1078a25. I am not confident whether Aristotle's usage here is a deviation from his usual usage or whether it provides evidence that the standard interpretation is in need of revision.

¹³ Cf., e.g., Annas, op. cit., pp. 148–49; J. J. Cleary, “Aristotle's Doctrine of Abstraction,” unpublished manuscript.

¹⁴ *Metaphysics* 1077b20–22, 1078a28–31; *Physics* 193b31–33, 194a9–11.

A difficulty for this interpretation might be thought to be presented by *Metaphysics* M3, 1077b31–34:

So since it is true to say without qualification not only that separable things exist but also that nonseparable things exist (e.g., that moving bodies exist), it is also true to say without qualification that mathematical objects exist and are as they are said to be.

It would be a mistake to interpret this sentence as asserting the existence of ideal objects. First, the expression “without qualification” (*haplōs*) certainly modifies “to say” (*legein* and *eipein* respectively) and not “to be” (*einai* or *estin*).¹⁵ For *Metaphysics* M2 closes with a categorical statement that mathematical objects “do not exist without qualification” (*ouk haplōs estin*, 1077b16). Rather, mathematical objects are supposed to exist in some qualified fashion. Indeed the task of *Metaphysics* M3 is to explicate the way in which mathematical objects can be said to exist. Second, that the sentence begins with “so since . . .” (*hōst’epi*) shows that the claim that separables and mathematical objects exist is the conclusion of the argument that preceded it. But that argument, as we have seen, denies that there are separate Platonic objects and affirms that geometry is about physical, perceptible magnitudes, though not considered *as* physical or *as* perceptible. Thus, for Aristotle, one can say truly that separable objects and mathematical objects exist, but all this statement amounts to—when properly analyzed—is that mathematical properties are truly instantiated in physical objects and, by applying a predicate filter, we can consider these objects as solely instantiating the appropriate properties. This interpretation is confirmed by the analogy Aristotle proceeds to make with the science of health (1077b34–1078a2). We can say without qualification that the science of health is about health; and we can disregard anything that is incidental to being healthy, for example, being pale. This does not mean that health exists independently of healthy men and animals; it means only that in studying health we can ignore everything that is irrelevant to that study.

So far Aristotle has argued that in studying geometry one need study only physical objects, not Platonic objects, though considered independently of their particular physical instantiation.

¹⁵ Cf. Ross, *op. cit.*, II, pp. 416–17.

The second major step in Aristotle's argument begins at 1078a17 where he says that if someone should postulate and investigate objects that are separated from incidental properties, he would not because of this be led to speak falsely. Why is this? Suppose, for example, we assume that there is an object c such that

$$(\forall G) (G(c) \leftrightarrow G(c \text{ qua } Tr)).$$

That is, we are assuming the existence of an object whose *only* properties are those that are logical consequences of its being a triangle. This is the assumption of a geometrical object, a triangle, separated from any material instantiation. Now suppose we should prove, via the proof of Euclid I-32, that c has interior angles equal to two right angles ($2R(c)$). Since by hypothesis we are guaranteed that the only properties of c are logical consequences of its being a triangle, we can argue from

$$2R(c)$$

to

$$(\forall x) (Tr(x) \rightarrow 2R(x)).$$

By universal instantiation we can infer

$$2R(b)$$

for any triangle b .

The reason one will not be led to speak falsely as a result of the fiction that there are separated objects, according to Aristotle, is that "the falsity is not in the premises" (1078a20-21). The analogy offered is with the case in which one draws a line (on a blackboard or in sand) and says for example, "Let the line AB be one foot long." Aristotle correctly sees that the line is drawn for heuristic purposes and is not a part of the proof. How is this analogous?

In the above proof one has assumed that there is a separated geometrical object c , but in fact from the point of view of the proof, there is no difference between c and any actual triangular

object b considered *as* a triangle— b *qua* Tr . For the properties one can prove to hold of c are precisely those one can prove to hold of c *qua* Tr , and it is dasy to show from the definition of the *qua*-operator that these are the same as the properties one can prove to hold of b *qua* Tr , regardless of the choice of b , provided only that it is a triangle. For heuristic purposes we may indulge in the fiction that c is a separated triangle, not merely some b considered *as* a triangle; this is a harmless fiction because the proof is indifferent as to whether c is *really* a separated triangle or just b considered as one. It is in this sense that the falsity does not enter into the premises.

The falsity would enter the premises if we were to concentrate not on properties that c must have, but on properties that c cannot be proved to have. For example, since

Triangle (x) $\not\vdash$ Isosceles (x)
 Triangle (x) $\not\vdash$ Scalene (x)
 Triangle (x) $\not\vdash$ Equilateral (x),

we have

\neg Isosceles (c *qua* Tr) & \neg Scalene (c *qua* Tr) & \neg Equilateral (c *qua* Tr),

and thus

\neg Isosceles (c) & \neg Scalene (c) & \neg Equilateral (c).

However,

Triangle (x) \vdash Isosceles (x) \vee Scalene (x) \vee Equilateral (x),

and so it follows that

Isosceles (c *qua* Tr) \vee Scalene (c *qua* Tr) \vee Equilateral (c *qua* Tr),

and thus

Isosceles (c) \vee Scalene (c) \vee Equilateral (c).

So it seems we can prove both that c is either scalene or equilateral or isosceles, and that it is neither scalene nor equilateral nor isosceles. The contradiction has arisen because the false assumption that c is a pure geometrical object has entered the premises: for it is essential to the proof itself that the only properties c has are the properties that follow from its being a triangle. The reason, for Aristotle, that one does not confront such problems in

geometry is that one is really only considering an actual physical object in abstraction from its particular physical instantiation. Though one cannot prove of *b qua Tr* that it is equilateral, one cannot correctly infer that *b* is not equilateral, only that it is not the case that *b considered as a triangle* is equilateral. The fiction that *c* is a separated geometrical object, a triangle *tout court*, is harmless only because we are concerned in geometry with the properties *c* does have, not with the properties *c* does not have. It is only if, ignorant of the foundations of one's mathematical practice, one takes the fiction of geometrical objects too seriously and begins to enquire philosophically into their nature that one runs into trouble. The analogy would be with someone who drew a line in the sand saying, "Let *AB* be one foot long," and then went on to say, "But *AB* is not a foot long; therefore it both is and is not the case that *AB* is a foot long." Similarly, one might object: "If something is a triangle, it has a definite shape; and if it has a shape, it must be colored; and yet a separated mathematical triangle is neither white, nor green, nor red . . ." Aristotle would respond that one is here running the risk of turning a harmless fiction into a dangerously misleading falsehood by importing the falsehood into the premises (cf. 1078a18–21). The fiction works, in part, due to the fact that as geometers we are concerned only with geometrical properties that follow from being a triangle. To think that the pure mathematical triangle *c* must be colored, on the ground that it has a definite shape, is, for Aristotle, to be confused about what one is doing when one talks about separated mathematical objects. For, as we have seen, there is one sense in which the separated mathematical triangle *c* does not have a definite shape: it is neither scalene nor isosceles nor equilateral. So, too, one must accept that it does not have a definite color. For in creating the harmless fiction of a separated mathematical triangle, one has abstracted from all considerations of any triangle's color.

That the postulation of separated objects is of heuristic value Aristotle is certain. The lesson of 1078a9–13 seems to be that the more predicates we can filter out, the more precise and simple our knowledge will be.¹⁶ The paradigm of a precise science was the articulated deductive system of geometry; as one moves toward a science of nature, one's knowledge becomes ever less

¹⁶Cf. Annas, op cit., p. 150; Ross, op. cit., II, p. 417.

precise. Astronomy could provide precise deductions of the planets' movements, but there were anomalies between the deductions and observed phenomena; and a general theory of dynamics was in much worse shape than astronomy (cf. 1078a12–13). Our knowledge is simpler because we have been able to filter out extraneous information. For if we have proved

$$(1) 2R(c),$$

then we know that we can infer that any particular triangle whatever has interior angles equal to two right angles. If by contrast we had proved only

$$(2) \text{Triangle } (b) \ \& \ F_1(b) \ \& \ \dots \ \& \ F_n(b) \ \vdash \ 2R(b),$$

then of course we could infer any instantiation of

$$(3) \text{Triangle } (x) \ \& \ F_1(x) \ \& \ \dots \ \& \ F_n(x) \ \vdash \ 2R(x);$$

but given any other triangle d such that

$$(4) \text{Triangle } (d) \ \& \ G_1(d) \ \& \ \dots \ \& \ G_k(d),$$

we would not be able to prove, on the basis of (2), that

$$(5) 2R(d).$$

For it is not clear on what properties of b the proof of (2) depends. However, (5) is an obvious consequence of (1) and (4): for it follows from (1) that

$$(6) 2R(d).$$

The postulation of separated geometrical objects enables us to attain knowledge that is more general. And it is through this general knowledge that one can discover the explanation (*aitia*) of why something is the case. For by abstracting one can see that the full explanation of a triangle's having the $2R$ property is that it is a triangle and not, say, that it is bronze or isosceles (cf. *An. Pst.* A5). In a limited sense, though, the abstract proof is unnecessary. For of any particular physical triangle d we can prove that it has interior angles equal to two right angles without first proving this for c : we could prove that d has the property directly. The proof *that* a physical object possesses a geometrical property via a proof that a pure geometrical object has the property is a useful, but unnecessary, detour. However, if we want to know *why* the

object possesses the property, the abstract proof is of crucial importance.

Thus it is that the best way of studying geometry is to separate the geometrical properties of objects and to posit objects that satisfy these properties alone (1078a21 ff.). Though this is a fiction, it is a helpful fiction rather than a harmful one: for, at bottom, geometers are talking about existing things and properties they really have (1078a28 ff.).

II

This interpretation of Aristotle's philosophy of geometry rests on the assumption that Aristotle thought that physical objects really do instantiate geometrical properties. One might wonder both whether this assumption is true and to what extent it needs to be true. There is, as we have seen, strong evidence in favor of the assumption: the passages from *Physics* B2 and *Metaphysics* M3 considered above repeatedly emphasize that the geometer studies physical objects, but not *as* physical objects. There is no mention that the physical objects do not possess geometrical properties, yet one would certainly expect Aristotle to mention this if he believed it. Further, throughout the Aristotelian corpus there are scattered references to bronze spheres and bronze isosceles triangles: there is no suggestion that these objects are not really spherical or really triangular.¹⁷

So in view of this *prima facie* evidence, the burden of proof shifts to those who believe that, for Aristotle, physical objects do not truly instantiate geometrical properties. Two passages are cited.¹⁸ I shall argue that neither need be construed as supporting the thesis that physical objects fall short of truly possessing geometrical properties. First, consider the passage from *Metaphysics* B2:

But on the other hand astronomy cannot be about perceptible magnitudes nor about this heaven above us. *For neither are perceptible lines such lines as the geometer speaks of (for no perceptible thing is thus*

¹⁷ I discuss this in detail below.

¹⁸ *Metaphysics* 997b35–998a6; 1059b10–12. Cf. Mueller, op. cit., p. 158; Annas, op. cit., p. 29.

straight or round: for a (physical) circle (e.g., a hoop) touches a straight edge not at a point, but as Protagoras used to say it did in his refutation of the geometers) nor are the movements and spiral orbits of the heavens like those which astronomy studies nor have points the same nature as stars. [Met. B2, 997b34–998a6; my emphasis]

One should not consider this passage in isolation from the context in which it occurs. *Metaphysics* B2 is a catalogue of philosophical problems (*aporiai*) presented from various points of view. None of it should be thought of as a presentation of Aristotle's considered view on the subject. It is rather a list of problems in response to which he will form his philosophical position. Immediately before the quoted passage Aristotle is putting forward the problem for the Platonists that the belief in Form-like intermediates involves many difficulties (997b12–34). The quoted passage can thus be read as an imagined Platonist's response: "Yes, the belief in intermediates is problematic, but, on the other hand, giving them up involves difficulties, too." Here it is an imagined Platonist speaking, and not Aristotle. So Aristotle is not endorsing Protagoras' view; he is presenting it as one horn of a dilemma that must be resolved. We have already seen Aristotle's proposed resolution; and it is one that involves asserting that some physical objects perfectly possess geometrical properties.

One might object: "Does this mean that Aristotle is committed to saying that the hoop *qua* circle touches the straight edge at a point?" The straightforward answer to this is, "Yes it does," but this is not as odd as it may initially appear. Protagoras' objection looks plausible because the hoops we tend to see are not perfectly circular, and so they obviously would not touch a straight edge—let alone the surfaces on which they actually rest, which are not perfectly straight—at a point. But Aristotle is not committed to saying that there are any perfectly circular hoops existing in the world. All Aristotle must say is: i) insofar as a hoop is a circle it will touch a straight edge at a point; ii) there are some physical substances that are circular. (Such circular substances need not be hoops.) Claim (i) is true: inasmuch as a hoop fails to touch a straight edge at a point, thus much does it fail to be a circle. And there is certainly evidence that Aristotle believed claim (ii). Of course, Aristotle thought that the stars were spheres and that they moved in circular orbits (*De Caelo* B11, 8). But there is also

evidence that he thought that even in the sublunary world physical objects could perfectly instantiate geometrical properties. In *Metaphysics* Z8 he says:

... just as we do not make the substratum, the brass, so we do not make the sphere, except incidentally, *because the bronze sphere is a sphere and we make that*. For to make a particular is to make a particular out of an underlying substrate generally. (I mean that *to make the brass round* is not to make the round or the sphere, but something else, i.e., *to produce this form in something different from itself*. For if we make the form we must make it out of something else; for this was assumed. For example, *we make a bronze sphere; and that in the sense that out of this, which is brass, we make the other which is a sphere*.) If, then, we also make the substrate itself, clearly we shall make it in the same way, and the process of making will regress to infinity. Obviously, then, the form also, or whatever we ought to call the shape present in the sensible thing, is not produced; for this is that which is made to be in something else either by art or by nature or by some faculty. *But that there is a bronze sphere, this we make. For we make it out of brass and the sphere; we bring the form into this particular matter and the result is a bronze sphere.* [1033a28–b10; my emphasis]¹⁹

Later, in Z10, he continues:

Those things that are the form and the matter taken together, e.g., the snub or *the bronze circle*, pass away into these materials and the matter is a part of them, but those things which do not involve matter but are without matter, of which the *logoi* are of the forms only, do not pass away, either not at all or at least not in this way. Therefore these materials are principles and parts of the concrete things, while of the form they are neither parts nor principles. And therefore the clay statue is resolved into clay and *the sphere into bronze* and Callias into flesh and bones, and *again the circle into its segments; for there are circles which are combined with matter. For "circle" is said homonymously both of the unqualified circle and of the individual circle because there is no special name for the individuals.* [1035a25–b3; my emphasis]²⁰

The individual circle is a physical object; one in which the form of a circle is imposed on some matter. There is no suggestion that

¹⁹ Cf. *Metaphysics* 1033b19–29 and 1036a2–6.

²⁰ Cf. also *Metaphysics* 1035a9–14: “The *logos* of a circle does not contain the *logos* of the segments . . . the segments on which the form supervenes; yet they are nearer the form than the bronze *when roundness is produced in bronze*” (my emphasis).

it is not really circular. This theme is developed further in the following chapter, Z11:

In the case of things which are found to occur in specifically different materials, *as a circle may exist in bronze or stone or wood*, it seems plain that these, the bronze or the stone, are no part of the essence of a circle, since it is found apart from them. Of all things which are not seen to exist apart, there is no reason why the same may not be true, *just as if all circles that had ever been seen were of bronze*; for nonetheless the bronze would be no part of the form; but it is hard to eliminate it in thought. [1036a31–b2; my emphasis]

Now at times it may seem as though Aristotle is saying that circles are not sensible objects. For example, he criticizes Socrates' analogy between an animal and a circle because it misleadingly encourages the thought that man can exist without his parts as the circle can exist without the bronze (1036b24–28). However, in such passages, Aristotle is not claiming that there are *no* sensible circles; he is claiming only that it is possible for a circle to exist independently of a particular physical instantiation. These are the intelligible mathematical circles which Aristotle contrasts with the perceptible circles of bronze and wood (1036a3–5). It is the task of *Metaphysics* M3–4 to explain the way in which such nonsensible circles exist.

The second passage cited to support the thesis that Aristotle did not believe that physical objects perfectly instantiate geometrical properties is two lines from *Metaphysics* K1:

. . . with what sort of thing is the mathematician supposed to deal? Certainly not with the things in this world, for none of these is the sort of thing which the mathematical sciences investigate. [*Met.* K1, 1059b10–12]

But again we must consider the context in which this statement occurs, in particular the larger passage of which it is a fragment:

. . . these thinkers [sc., Academic Platonists] place the objects of mathematics between the Forms and perceptible things as a kind of third set of things apart from the Forms and from the things in this world; but there is not a third man or horse besides the ideal and the individuals. If, on the other hand, it is not as they say, with what sort of thing must the mathematician be supposed to deal? Certainly not with the things in this world; for none of these is the

sort of thing which the mathematical sciences investigate. [*Met.* K1, 1059b6–12]

Now it should be clear that we are only being faced with the very same dilemma already presented to us in *Metaphysics* B2. Indeed, *Metaphysics* K1 is only a recapitulation of the *aporiai* of *Metaphysics* B2, 3. Again, we should read *Metaphysics* 1059b10–12 as the Academic Platonist's response to the objection that the postulation of intermediates involves many difficulties. Further, the claim that the things in this world are not the sort of thing which the mathematical sciences investigate need not even be read as a claim that, for example, a bronze sphere could not be perfectly spherical. Rather, it could be the claim that when a geometer considers a sphere he does not consider it as made up of bronze or any other matter. Aristotle could then see himself as providing a solution superior to the Platonist's: for he can explain this horn of the dilemma without having to resort to postulating problematic intermediates. He could thus think of himself as *dissolving* the dilemma, rather than accepting either horn. The great advantage of being able to dismiss these two passages is that it becomes immediately evident how Aristotle thought geometry could apply to the physical world.

There does, however, remain room for skepticism. Even if one grants that there are, for example, perfect spheres in the physical world, must there be perfect physical instantiations of every figure the geometer constructs? Surely, the skeptic may object, Aristotle should not commit himself to there being, for example, perfectly triangular bronze figures. And, the skeptic may continue, even if there were perfectly triangular physical objects, there are no physical instantiations of the more complex figures which a geometer constructs when he is proving a theorem. I think it is clear how Aristotle would respond. At the end of *Metaphysics* Θ9 he says:

It is by an activity also that geometrical constructions (*ta diagrammata*) are discovered; for we find them by dividing. If they had already been divided, the constructions would have been obvious; but as it is they are present only potentially. Why are the angles of a triangle equal to two right angles? Because the angles about one point are equal to two right angles. If, then, the line parallel to the side had been already drawn upwards, it would have been evident

why ⟨the triangle has such angles⟩ to anyone as soon as he saw the figure. . . . Obviously, therefore, the potentially existing things are discovered by being brought to actuality; the explanation is that thinking is an actuality. . . . [1051a21–31]

The geometer, Aristotle says, is able to carry out geometrical constructions in thought: the thinking which is an activity makes actual the construction which existed only potentially before the thinking occurred.²¹ Thus we can make sense of Aristotle's claim:

That is why the geometers speak correctly: they talk about existing things and they really do exist—for what exists does so in one of two senses, in actuality or materially. [*Met.* M3, 1078a28–31]

The word “materially” (*hulikōs*) connotes both the “matter” from which the construction is made and also the potentiality, associated with matter, of the geometrical figure before the activity of thought.²²

Has, then, Aristotle severed the tie between pure mathematics and the physical world? Does the geometer contemplate pure mathematical objects that are not in any way abstractions from the physical world? I don't think so. For to retain the link between geometry and the physical world, Aristotle need only maintain that the elements of a geometrical construction are abstractions from the physical world. Not every possible geometrical construction need be physically instantiated. In Euclidean geometry, constructions are made from straight lines, circles, and spheres. We have already seen the evidence that Aristotle thought that there were perfectly circular physical objects. Evidence that he thought there were also physical objects with perfectly straight edges is in the *De Anima*:

If, then, there is any of the functions or affections of the soul which is peculiar to it, it will be possible for it to be separated from the body. But if there is nothing peculiar to it, it will not be separable, but it will be like the straight, to which, *as* straight, many properties belong, e.g., *it will touch a bronze sphere at a point*, although the straight

²¹ I further discuss the role of mental activity in Aristotle's philosophy of mathematics in “Aristotelian Infinity,” *Proceedings of the Aristotelian Society*, 80 (1979/80).

²² Cf. also, e.g., *Metaphysics* 1048a30–35.

if separated will not so touch; for it is inseparable, if it is always found in some body. [403a10–16; my emphasis]

What Aristotle takes to be touching a bronze sphere at a point is a *physical* straight edge, for when the straight line has been abstracted it cannot touch a physical object at all.²³ Since a geometrical triangle can be constructed in thought from straight lines, Aristotle does not have to say that a particular bronze figure *d* is perfectly triangular. If it is—and given that he thought there could be a physical straight edge, there is no reason for him to deny that there could be a physical triangle—then one can apply the *qua*-operator to it and proceed to prove theorems about it *qua* triangle. If it is not, then the properties that have been proved to hold of triangles will hold of it more or less depending on how closely it approximates being a perfect triangle. We could then relax our claim that *d qua F* is *G*, and say only that *d* insofar as it is *F* is *G*.

The important point is that direct links between geometrical practice and the physical world are maintained. Even in the case where the geometer constructs a figure in thought, one which perhaps has never been physically instantiated, that figure is constructed from elements which are direct abstractions from the physical world. Otherwise it will remain a mystery how, for Aristotle, geometry is supposed to be applicable to the physical world. Certainly it is a mistake to treat Aristotle's few remarks about intelligible matter as providing either the basis of a Platonic epistemology or a prototype of the form of outer intuition.²⁴ He specifically says that mathematical objects have intelligible matter,²⁵ and in these contexts it is used to do justice to the nature of mathematical thinking: that is, when one carries out a geometrical proof it seems as though one has a particular object in mind. To prove a general theorem about triangles, it seems as though one chooses an arbitrary particular triangle on which one

²³Cf. also *De Anima* 431b15–17; 432a3–6. Note Sextus Empiricus' report: "Aristotle, however, declared that the length without breadth of the Geometers is not inconceivable ("For in fact we apprehend the length of a wall without having a perception of the wall's breadth") . . ." (*Adversus Mathematicos*, ix, 412).

²⁴ Cf. *Metaphysics* 1036a2–12; 1036b32–1037a5; 1045a33–35. See also 1059b14–16; 1061a26–30.

²⁵ *Metaphysics* 1036a9–12; 1037a2–5; cf. 1059b14–16; 1061a28–35.

performs a construction.²⁶ Thus it does not seem that one merely has the form of, for example, triangularity in mind: intelligible matter is invoked to account for the fact that we are thinking about a particular object.

Aristotle's philosophy of mind, however, falls significantly short of being Platonic. For Aristotle does not have to postulate the existence of an object that does not exist in the physical world to be the object of thought; he merely has to explain how we think about objects that do exist in the world.²⁷ The problem for Aristotle is the nature of thinking in general, not the existence of a special type of mathematical object to think about. It is true that we can both perceive a sphere and think about it. In fact, we can think about it in abstraction from the fact that it is composed, for example, of bronze. We can even perform mental constructions and form a figure that we have perhaps never perceived. But even in this case we are doing no more than constructing a figure in thought from elements that are direct abstractions from the physical world.

One might wonder whether, on this interpretation, it follows that perceptible objects have intelligible matter. The answer is that they have intelligible matter insofar as they can be objects of thought rather than perception: that is, it is the object one is thinking about that has intelligible matter. The evidence for this

²⁶ Cf. Euclid I-32; and the interpolation after Euclid III-3. See also *Posterior Analytics* 94a24-35; *Metaphysics* 1051a26; T.L. Heath, *Mathematics in Aristotle* (Oxford: Clarendon Press, 1970), pp. 31-39, 71-74, 216-17.

²⁷ A critic of Aristotle might say that Aristotle had not provided a significant advance on Plato, since Aristotle often says, e.g., that knowledge is of the universal (cf., e.g., *Metaphysics* 1003a13-15, 1060b19-21, 1086b5-6, 32-37; *De Anima* 417b22-23; *Nicomachean Ethics* 1140b31-32, 1180b15-16. See also *Posterior Analytics* 87b28-88a17.) and that a proper scientific proof will be of the universal (cf. *Posterior Analytics* 85b15 ff.). However, the claim that "knowledge is of the universal" admits of widely different interpretations, depending on what one takes Aristotle's theory of the universal to be. I am inclined to interpret his theory of universals and of knowledge of the universal in a strongly non-Platonic fashion; but I cannot defend that interpretation here. In this paper I am not trying to explicate fully Aristotle's theory of knowledge, nor am I claiming that it is completely unproblematical. I wish only to make the weaker claim that mathematical knowledge is no more problematical for Aristotle than knowledge of anything else. So if we grant Aristotle that we can have knowledge of the properties of a physical object, it follows that we can have mathematical knowledge.

is Aristotle's claim that intelligible matter is "the matter which exists in perceptible objects but not *as* perceptible, for example, mathematical objects" (1036a11–12).

III

Aristotle's philosophy of arithmetic differs significantly from his philosophy of geometry, though Aristotle is not especially sensitive to this difference. The predominant mathematics of ancient Greece was geometry, and thus it is not surprising that his philosophy of mathematics should be predominantly a philosophy of geometry.

The main obstacle preventing Aristotle from giving a successful account of arithmetic is that number is not a property of an object.²⁸ Thus one cannot legitimately think of a number as one of the various properties of an object that can be separated from it in thought. One can, however, see Aristotle grappling with the problem:

The best way of studying each thing would be this: to separate and posit what is not separate as the arithmetician does, and the geometer. A man is one and indivisible *as* a man, and the arithmetician posits him *as* one indivisible; the geometer, on the other hand, studies him neither *as* a man nor *as* indivisible, but *as* a solid object. For clearly properties he would have had even if he had not been indivisible can belong to him irrespective of his being indivisible or a man [*aneu toutōn*]. [*Met.* M3, 1078a22–28]

Aristotle's position is, I think, as follows. Substances, that is, members of natural kinds, carry, so to speak, a first-level predicate with them as their most natural form of designation. A man is first and foremost a man. Thus when one considers a man *as a man*, one is *not* abstracting one of the many properties a man may possess from the others; one is rather selecting a unit of enumeration. In Fregean terms, one is bringing objects under a first-level concept. Elsewhere Aristotle allows that the number of sheep, of men, and of dogs may be the same even though men, sheep, and dogs differ from each other.²⁹ The reason is that we are given the individual

²⁸ Cf. G. Frege, *The Foundations of Arithmetic*, (Oxford: Blackwell, 1968).

²⁹ Cf., e.g., *Physics* 220b8–12; 223b1–12; 224a2–15.

man (or sheep or dog) as a unit, and the enumeration of each group yields the same result. The arithmetician posits a man *as* an indivisible because he posits him as a unit and as a unit he is treated as the least number (of men).³⁰

Thus Aristotle uses the *as*-locution for two distinct purposes. In geometry it is used to specify which property of a physical object is to be abstracted from others and from the matter. In arithmetic it is used to specify the unit of enumeration. It is easy to see how in a pre-Fregean era these two uses could be run together. Both uses could be loosely expressed as “considering an x in respect of its being an F .” In one case we can consider a bronze sphere in respect of its being a sphere; in the other we can consider Socrates in respect of his being a man. Indeed, in both cases one can be said to be “abstracting”: in the former one abstracts from the fact that the sphere is bronze; in the latter one abstracts from the fact that one man is many-limbed, snub-nosed, etc. But to run these two uses together may be misleading. For in the former case we are picking out one of the object’s many properties and separating it in thought. In the latter case we are picking out the object itself, under its most natural description, and specifying it as a unit for counting. If one thinks, that, *strictly speaking*, abstraction is the separation of a property of the object, then there is in the case of arithmetic no abstraction at all. Aristotle would have had a more difficult time conflating these uses if he had not instinctively switched to natural kind terms when discussing arithmetic. For suppose Aristotle had asked us to consider a sphere *as a sphere* (*hē sphaira hēi sphaira*): there would be no way of knowing on the basis of the locution whether we were to treat the sphere as a unit in counting spheres or to consider the spherical aspect of a sphere in abstraction from its other properties.

IV

Since Frege, philosophies of mathematics that can be labelled “abstractionist” have been in bad repute. It is, however, a mistake to tar Aristotle with Frege’s brush. Frege was attacking psychologism, the attempt to reduce logic and mathematics to laws of

³⁰ Cf. *Physics* 220a27–32; *Metaphysics* 1092b19.

empirical psychology. He was especially critical of the idea that number was anything subjective or dependent for its existence on the existence of some inner mental process.³¹

Frege treats abstraction as a deliberate lack of attention:

We attend less to a property and it disappears. By making one characteristic after another disappear, we get more and more abstract concepts Inattention is a most efficacious logical faculty; presumably this accounts for the absentmindedness of professors. Suppose there are a black and a white cat sitting side by side before us. We stop attending to their color and they become colorless, but they are still sitting side by side. We stop attending to their posture and they are no longer sitting (though they have not assumed another posture) but each one is still in its place. We stop attending to position; they cease to have place but still remain different. In this way, perhaps, we obtain from each one of them a general concept of Cat. By continued application of this procedure, we obtain from each object a more and more bloodless phantom. Finally we thus obtain from each object a *something* wholly deprived of content; but the *something* obtained from one object is different from the *something* obtained from another object—though it is not easy to say how.³²

Frege's point, made repeatedly with varying degrees of humor and sarcasm, is that if we "abstract" from all the differences between objects, then it will be impossible to count them as different objects.

One must distinguish two strands of thought that run together through Frege's criticisms. First, there are the attacks on psychologism, on treating "abstraction" as a deliberate lack of attention. Second, there is his criticism of treatments of number as a first-level concept, of number as a property of an object. Since number is not a property of an object, it is not there to be abstracted even if abstraction were a legitimate process. As one reads Frege's critique of his predecessors in the *Foundations of Arithmetic*, one can see that their problem lies mainly in their assumption that number is a first-level concept, not in whether or not they are abstractionists.

³¹ Cf. Frege, op. cit., pp. 33–38.

³² From Frege's review of Husserl's *Philosophie der Arithmetik*, in *Translations from the Philosophical Writings of Gottlob Frege*, ed. P. Geach and M. Black (Oxford: Blackwell, 1970), pp. 84–85.

If, however, one considers genuine properties of objects, in particular, geometrical properties actually possessed by physical objects, then it becomes clear that not all forms of abstraction need be treated as psychologistic or, indeed, as forms of inattention. For Aristotle, abstraction amounts to no more than the separation of one predicate that belongs to an object and the postulation of an object that satisfies that predicate alone. This separation may occur “in thought” (*Phys.* 193b34), but this is no more damning than admitting that one carries out a geometrical proof or arithmetical calculation “in thought.” One is carrying out a determinate procedure and there is no irremediably subjective element.

V

Of course Aristotle’s philosophy of mathematics does have its limitations, but what is remarkable is that, even from a contemporary perspective, it retains certain strong virtues. The limitations are obvious enough. In arithmetic we are given only a means of selecting a unit for enumeration. In geometry Aristotle’s method depends on there being actual physical objects which possess all the relevant properties with which we reason geometrically. There are two features of mathematical experience to which Aristotle’s theory does not seem to do justice. First, much of mathematics, for example set theory, cannot easily be thought of as an abstraction from any aspect of physical experience. Second, there is the plausible belief that mathematical theorems are true irrespective of whether there is any physical instantiation of them. Under scrutiny, however, this belief turns out to be less categorical than one might have expected. For our belief in the truth of a mathematical theorem may not depend on the actual existence of a bronze triangle, but it is inextricably linked to our belief in the applicability of mathematics to the world. Euclid I-32, once thought to be an a priori truth, is no longer even thought to be true. The reason is that it is now believed that physical triangles, if there were any, would not have interior angles equal to two right angles. Euclid I-32 may be a consequence of the Euclidean axioms—and thus we can make the

limited claim that the theorem is true *of* triangles in Euclidean space—but one of the axioms which enters essentially into the proof is thought to be false of the physical world. So while one may believe a theorem true while remaining agnostic on the question, say, of whether there are any physical triangles, one must believe that a triangle is a physical possibility and that the theorem truly describes a property it would have.

The virtues of Aristotle's philosophy of mathematics are most clearly seen by comparing it with the philosophy of mathematics advocated by Hartry Field in *Science Without Numbers*.³³ Field argues that to explain the applicability of mathematics one need not assume it to be true. One need only assume that one's mathematical theory (M) is a *conservative extension* of one's physical theory (P). Suppose S is a sentence using solely terms of physical theory. Then to say that M is a conservative extension of P is to say that

if $M, P \vdash S$, then $P \vdash S$.

That is, any sentence of physical theory which can be proved with the aid of mathematics can be proved without it. The invocation of mathematics makes the proof simpler, shorter, and more perspicuous, but it is in principle eliminable. And for mathematics to be a conservative extension of science, it need not be true; it need only be consistent.³⁴

Aristotle treated geometry as though it were a conservative extension of physical science. If, as we have seen, one wants to prove of a particular bronze isosceles triangle b that it has interior angles equal to two right angles, one may “cross” to the realm of pure geometrical objects and prove the theorem of a triangle c . The $2R$ property has been proved to hold of c in such a way that it is evident that it will hold of any triangle, so one can then “return” to

³³ Hartry Field, *Science Without Numbers* (Oxford: Blackwell, 1980).

³⁴ It is not generally true that the consistency of a theory will guarantee that it will apply conservatively to the theory to which it is adjoined. But Field makes ingenious use of the fact that mathematics can be modeled in set theory to show that in the special case of mathematics, consistency does suffice for conservativeness. See Field, *op.cit.*, Chapter 1 and Appendix. To see that consistency does not in general imply conservativeness, consider two consistent theories T_1 , which has P as a theorem, and T_2 , which has $P \supset Q$ as a theorem. Even if the union of T_1 and T_2 is consistent, $T_1 + T_2$ is not a conservative extension of either T_1 or T_2 , since in $T_1 + T_2$, Q is a theorem, although it is not a theorem of either T_1 or T_2 .

the physical world and conclude that b has the $2R$ property. The “crossing” was not, however, strictly speaking necessary: one could have proved directly of the bronze isosceles triangle b that it had interior angles equal to two right angles. The reason why the “crossing” is valuable, though, is that one thereby proves a general theorem applicable to all triangles rather than simply proving that a certain property holds of a particular triangle. Thus if we let AP stand for Aristotle’s physical theory, AG for Aristotelian geometry, and S for an arbitrary sentence in the language of AP , Aristotle would have allowed that

if $AP, AG \vdash S$, then $AP \vdash S$.

Geometry, for Aristotle, was a conservative extension of physical theory.

The great virtue of Aristotle’s account is that Aristotle also takes great pains to explain how mathematics can be true. A conservative extension of physical theory need not merely be consistent; it can also be true. Aristotle tries to show how geometry and arithmetic can be thought of as true, *even though* the existence of separated mathematical objects, triangles and numbers, is a harmless fiction.³⁵ That is, he tries to show how mathematical statements can be true in a way which does not depend on the singular terms having any reference or the quantifiers ranging over any separated mathematical objects. The key to explaining the truth of a mathematical statement lies in explaining how it can be useful. Aristotle considered the truths of geometry to be useful because there are clear paths which lead one from the physical world to the world of geometrical objects. There may be no purely geometrical objects, but they are a *useful* fiction, because they are an obvious abstraction from features of the physical world. Merely to say that an arbitrary theory T is a conservative extension of our theory of the physical world (P) will not

³⁵ Let us use the word “Platonist” to describe the position in the philosophy of mathematics held by Plato and his followers in the Academy. Let us use “platonist” to describe anyone who believes that mathematical statements are true in virtue of the existence of abstract objects which exist outside space and time. (Kurt Gödel is an example of a platonist.) Finally, let us say that a “mathematical realist” is someone who believes that mathematical statements are determinately true or false independently of our knowledge of them. Then one can say that Aristotle defends a form of mathematical realism while denying both Platonism and platonism.

explain the usefulness or applicability of T . It would be easy to formulate a consistent theory T , prove that $P + T$ is a conservative extension of P , and show that T is of no use whatsoever in deriving consequences about the physical world. It is precisely because mathematics is so richly applicable to the physical world that we are inclined to believe that it is not merely one more consistent theory that behaves conservatively with respect to science, but that it is true.

What is needed is a bridge between the physical world and the world of mathematical objects, similar to the bridge Aristotle provided between bronze triangles and geometrical triangles, that will enable us to see how we can cross to the world of mathematical objects and return to the physical world. This bridge would explain both the sense in which mathematics is an abstraction of the physical world and why it is applicable.

This bridge, Field suggests, is supplied by Hilbert's *representation theorem* for Euclidean geometry.³⁶ The proof of the representation theorem shows that given any model of Hilbert's geometrical axioms, there will be functions from points in space into the real numbers which satisfy conditions for a distance structure. Given that, one can show that the standard Euclidean theorems are equivalent to theorems about relations between real numbers. So if one thinks of models of space as being abstract, there is a two-stage process of moving from the physical world to the mathematical. The first stage is Aristotle's, where one moves from the physical world to a Euclidean model of space; the second is where one moves from the Euclidean model to a model of Euclidean space in the real numbers. The homomorphic functions would then provide the second span of the bridge and Aristotelian abstraction would provide the first. Or one could just take physical space as the model for the axioms (assuming that the axioms are true of physical space), and then one needs only the homomorphic functions as a bridge.

This example is specific to geometry, but it contains the key to a general theory of the applicability of mathematics. For

³⁶Field, *op. cit.*, Chapter 3. Cf. David Hilbert, *Foundations of Geometry* (Lasalles: Open Court, 1971). Of course, since Aristotle, we have learned that to formalize geometry successfully we need more axioms than geometers of Aristotle's day were aware of, particularly about the relation of betweenness, and that geometry has undergone a thorough arithmetization.

mathematics to be applicable to the world it must reproduce structural features that are found (at least to some approximate degree) in the physical world. Moreover, there must be a bridge by which we can cross from the structural features of the world to the mathematical analogue, and then return to the physical world. Consider a simple example from set theory. To say that arithmetic can be modelled in set theory is to say that there exist at least one function f_0 that maps the natural numbers one-one into sets and functions f_1 and f_2 which map pairs of sets into sets such that

$$x + y = z \leftrightarrow f_1 (f_0 (x), f_0 (y)) = f_0 (z)$$

and

$$x \cdot y = z \leftrightarrow f_2 (f_0 (x), f_0 (y)) = f_0 (z).$$

The functions f_1 and f_2 impose a structure on sets relevantly analogous to that of the natural numbers, and f_0 provides a bridge between the natural numbers and the universe of sets. There may, of course, be many triples $\langle f_i, f_j, f_k \rangle$ which satisfy these constraints.³⁷ What remains to be supplied is a bridge between natural numbers and the physical world. This bridge was successfully specified by the logicians in their otherwise unsuccessful attempt to reduce mathematics to logic. A number n is related to other numbers in ways which are intimately linked to the manner in which n -membered sets of durable physical objects are related to disjoint sets of various cardinality. The union of two disjoint two-membered sets of durable physical objects usually is a four-membered set, and the arithmetical truth “ $2 + 2 = 4$ ” reflects this fact. Of course, physical objects may perish or coalesce with others—thus the “usually”—and one of the virtues of arithmetic is that in crossing to the realm of numbers one can abstract from this possibility.

Thus, to explain the usefulness and applicability of mathematics we have had to follow Aristotle and appeal far more strongly to the existence of a bridge between the physical world and the world of mathematical objects than we have to the fact that mathematics is a conservative extension of science. The conservativeness of mathematics was invoked by Field to explain why

³⁷ Cf. Paul Benacerraf, “What Numbers Could Not Be,” *Philosophical Review*, 74 (1965).

we need not think of mathematics as true, but only as consistent. But to explain why this particular consistent theory rather than others is *useful*, we have had to rely rather heavily on the existence of bridges and thus, I think, to *reimport* the notion of truth. For, in an Aristotelian spirit, one can allow that “ $2 + 2 = 4$ ” is true without having to admit that there exist numbers in a Platonic realm outside of space and time. That there must exist bridges between the physical world and those portions of mathematics which are applicable to it implies that the mathematics must reproduce (to a certain degree of accuracy) certain structural features of the physical world. It is in virtue of this accurate structural representation of the physical world that applicable mathematics can fairly be said to be true.

The crucial contrast between Aristotle and Hartry Field is as follows. For Aristotle, mathematics is true, not in virtue of the existence of separated mathematical objects to which its terms refer, but because it accurately describes the structural properties and relations which actual physical objects do have. Talk of nonphysical mathematical objects is a fiction, one that may be convenient and should be harmless if one correctly understands mathematical practice. Field agrees with Aristotle that there are no separated mathematical objects, but thinks that for that reason alone mathematics is not true. From an Aristotelian perspective, Field looks overly committed to the assumptions of referential semantics: in particular, to the assumption that the way to explain mathematical truth is via the existence of mathematical objects. One can understand how mathematics can be true, Aristotle thinks, by understanding how it is applicable.

Of course, as we now realize in contrast to Aristotle, not all mathematics need be applicable. And where portions of mathematics are not applicable, there is no compelling reason to think them true.³⁸

³⁸ For example, the assumption that there exists a measurable cardinal is not thought to enhance the applicability of set theory to the physical world. (Cf. F. Drake, *Set Theory: An Introduction to Large Cardinals* (Amsterdam: North Holland, 1974), Chapter 6.) Here all that is important is that one can consistently augment the standard axioms of set theory with an axiom asserting the existence of a measurable cardinal. From the point of view of applicability one could as well have added an axiom asserting the nonexistence of a measurable cardinal. Thus, under the assumption that set theory + “there exists a

Where is one to draw the line between the truths of mathematics and the parts of mathematics that are consistent with them, but neither true nor false? Nowhere: for there is no demarcation between applicable and nonapplicable mathematics that can be made with any certainty. One cannot determine a priori that a portion of mathematics is not applicable. It is conceivable, though unlikely, that we should discover that the world is sufficiently large and dense that we need a large cardinal axiom to describe it.³⁹ But this does not mean that we should treat all of mathematics as true or all of it as merely consistent. That we cannot draw a distinction precisely does not mean that there is no distinction to be made: it means only that any suggested boundary will remain conjectural and subject to revision. Thus, the question of how much of mathematics is true can only be answered a posteriori.

Though not Aristotle's, this philosophy of mathematics is Aristotelian. One of its virtues is that it does provide a harmonious account of truth and knowledge. Of those portions of mathematics that are not true, the question of knowledge does not arise. Of those portions that are true, there exist bridges of a fairly direct sort between the physical realm and the mathematical. And it is in virtue of our understanding of how these bridges link the mathematical and the physical that we can be said to know mathematical truths. Mathematics, according to Aristotle, studies the physical world, but not *as* physical: I should like to think that this approach to the philosophy of mathematics provides an explication of that insight.⁴⁰

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measurable cardinal" has no physical consequences that set theory does not have on its own, there is no reason to say that the measurable cardinal axiom is true or false.

³⁹ For example, if the cardinality of the physical continuum was found to equal not \aleph_1 , but the first measurable cardinal μ .

⁴⁰ I would like to thank M. F. Burnyeat, G. E. R. Lloyd, M. Schofield, T. J. Smiley, and the editors of the *Philosophical Review* for criticisms of an earlier draft.